The Definite Integral.

Example: Distance = Velocity · Time

If velocity is constant.

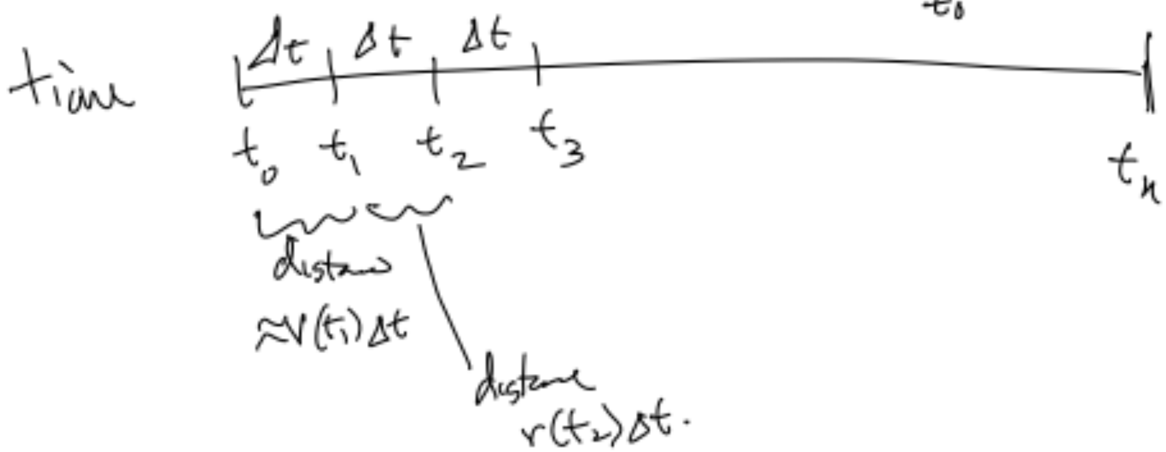
If velocity is not constant

Velocity = $v(t)$ a function of time

then

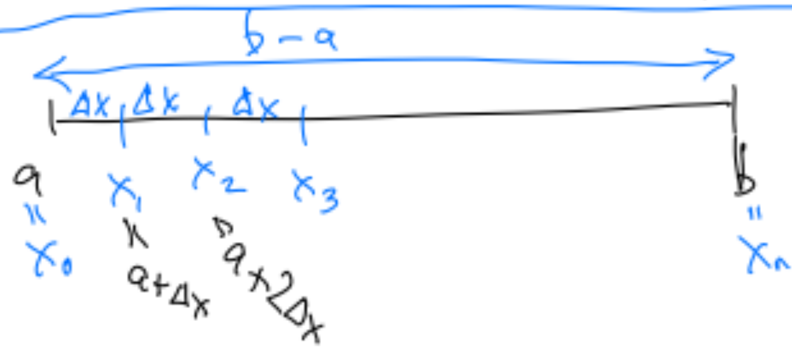
$$\text{Distance} = \lim_{n\to\infty} \sum_{j=1}^{n} v(t_j)\Delta t$$

$$= \int_{t_0}^{t_n} v(t)\,dt$$



In <u>general</u>: If f is a function on the interval $[a,b]$ (we assume f is continuous except at a finite # of discontinuities),

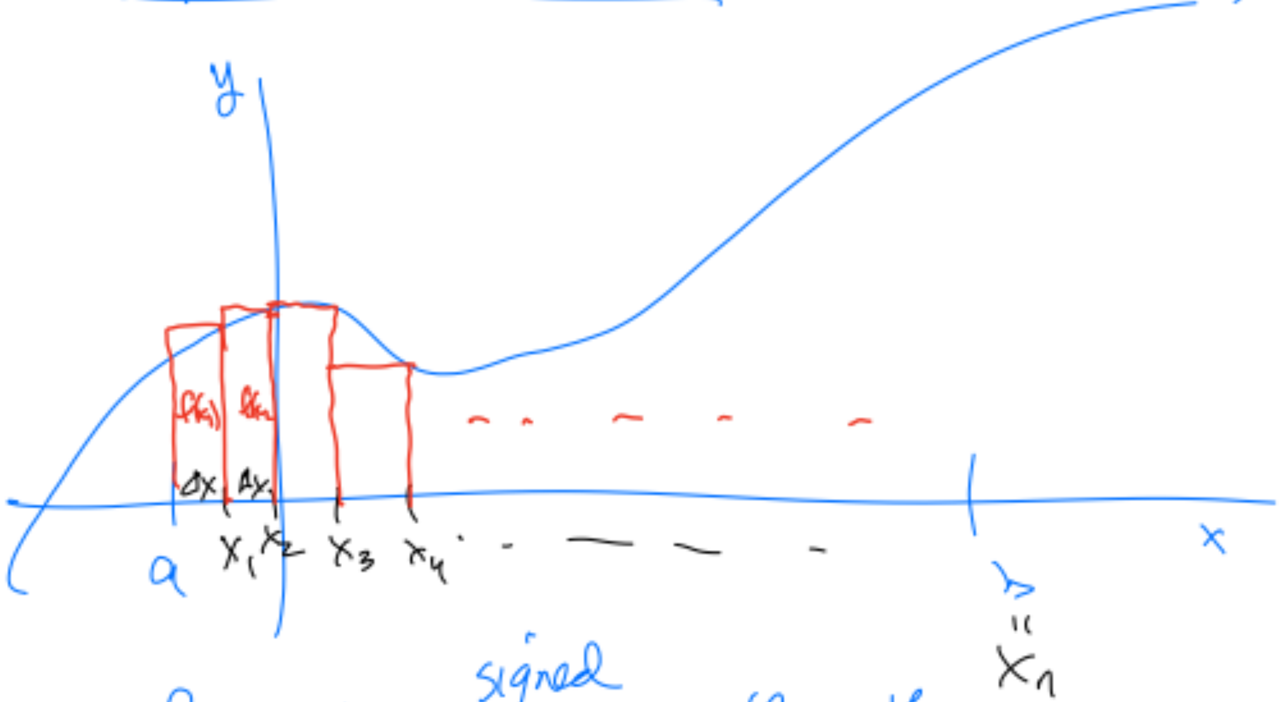
$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x, \quad \text{where } \Delta x = \frac{b-a}{n} \text{ and } x_k = a + k\Delta x}$$



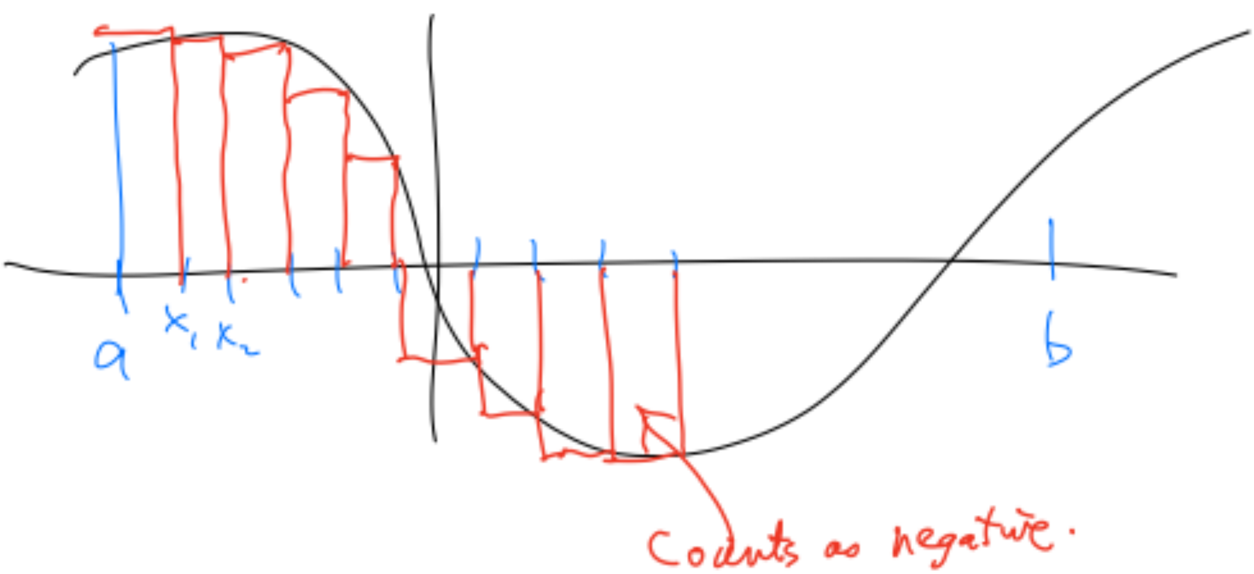
$\int_a^b f(x)\,dx$ = "the integral from a to b of f(x) dx"

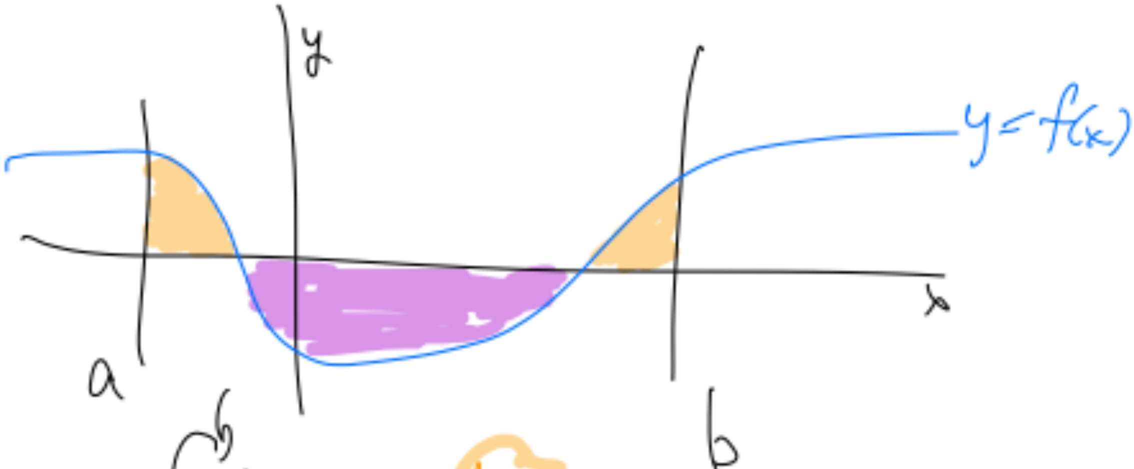
In most cases, we can't calculate the integral exactly — you have to use a computer.

Graphical Interpretation:



$f(x_j)\Delta x$ = signed area of the $j^{\underline{th}}$ rectangle under the curve, where the height is given by the right hand side value of the function.





$$\int_a^b f(x)\,dx = \text{Area} - \text{Area}$$

= Signed area under the curve $y=f(x)$ from $x=a$ to $x=b$.

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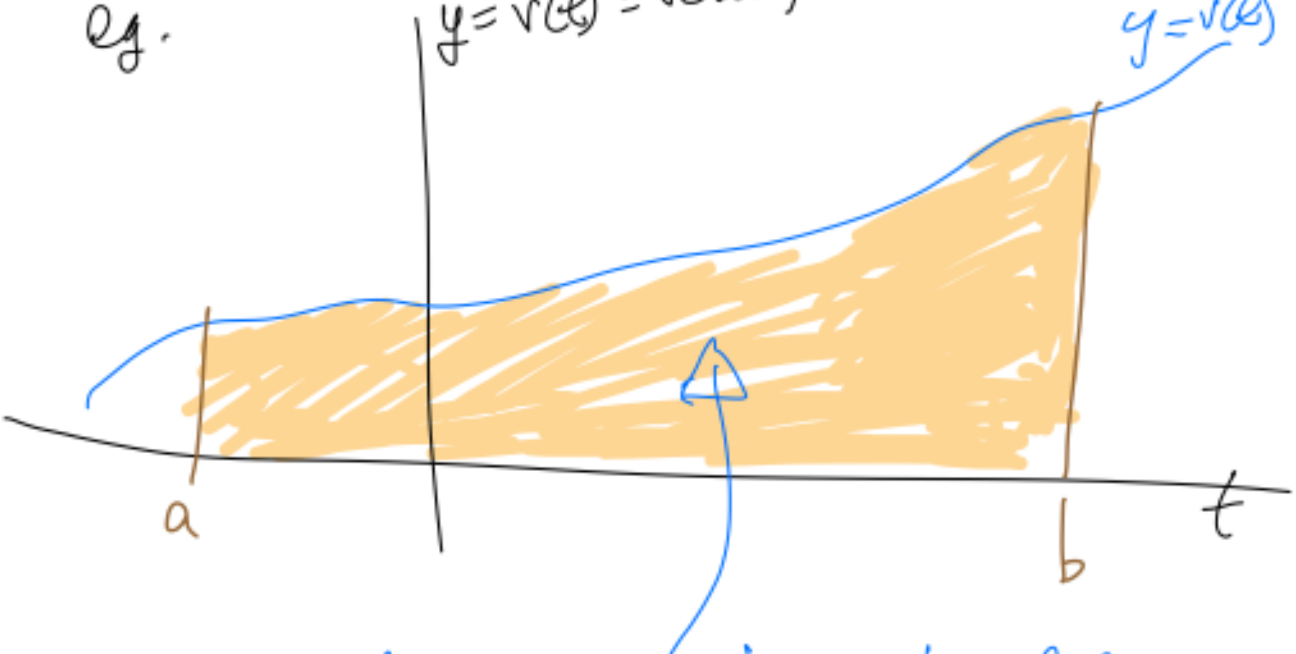
eg. $y = v(t)$ = velocity at time t



This area = distance traveled between $t=a$ & $t=b$.

Example of Computing $\int_a^b f(x)\,dx$ exactly.

Compute $\int_{-1}^{2} (x^2-3x)\,dx$.

Solution:

$$\int_{-1}^{2}(x^2-3x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\,\Delta x,$$

where $\Delta x = \frac{b-a}{n}$, $x_k = a + k\Delta x$.

In our case, $f(x) = x^2 - 3x$

$a=-1$, $b=2$

$$\Delta x = \frac{b-a}{n} = \frac{3}{n}.$$

$$x_k = -1 + \frac{3k}{n}$$

$$f(x_k) = \left(-1+\frac{3k}{n}\right)^2 - 3\left(-1+\frac{3k}{n}\right)$$

$$= 1 + -\frac{6k}{n} + \frac{9k^2}{n^2} + 3 - \frac{9k}{n}$$

$$\Rightarrow f(x_k) = \frac{9k^2}{n^2} - \frac{15k}{n} + 4.$$

$$\int_{-1}^{2} (x^2 - 3x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \left[\frac{9k^2}{n^2} - \frac{15k}{n} + 4\right]\frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left[\frac{27k^2}{n^3} - \frac{45k}{n^2} + \frac{12}{n}\right]$$

Recall: $\sum_{k=1}^{n} 1 = n$, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$,

$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

$$= \lim_{n\to\infty} \frac{27}{n^3}\left(\sum_{k=1}^{n} k^2\right) - \frac{45}{n^2}\left(\sum_{k=1}^{n} k\right) + \frac{12}{n}\sum_{k=1}^{n} 1$$

$$\Rightarrow \int_{-1}^{2}(x^2-3x)\,dx = \lim_{n\to\infty}\left(\frac{\cancel{27}\ 9}{\cancel{n^3}\ n^2}\right)\left(\frac{\cancel{n}(n+1)(2n+1)}{\cancel{6}\ 2}\right) - \left(\frac{45}{\cancel{n^2}\ n}\right)\left(\frac{\cancel{n}(n+1)}{2}\right) + \frac{12}{\cancel{n}}\cdot\cancel{n}$$

$$= \lim_{n\to\infty} \frac{9(n+1)(2n+1)}{2n^2} - \frac{45(n+1)}{2n} + 12$$

$$\frac{(n+1)(2n+1)}{n^2} = \frac{n^2\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{n^2} \to 2$$

$$\frac{n\left(1+\frac{1}{n}\right)}{n} \to 1$$

$$= \frac{9}{2}\cdot 2 - \frac{45}{2} + 12$$

$$= \frac{18-45+24}{2} = \boxed{-\frac{3}{2}}.$$

$\boxed{\text{Example}}$ Compute

$$\int_0^2 2^{-x}\,dx$$

Solution:

"Right Riemann Sum"

$$\int_0^2 2^{-x}\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$, $x_k = a + k\Delta x$

In our case, $a=0$, $b=2$

$$\Delta x = \frac{b-a}{n} = \frac{2}{n}$$

$$x_k = a + k\Delta x = \frac{2k}{n}.$$

$$f(x_k) = 2^{-\frac{2k}{n}}$$

$$\therefore \int_0^2 2^{-x}\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \left(2^{-\frac{2k}{n}}\right) \cdot \frac{2}{n}$$

$$= \lim_{n\to\infty} \frac{2}{n}\sum_{k=1}^{n} 2^{-\frac{2k}{n}}$$

$$= \lim_{n\to\infty} \frac{2}{n}\underbrace{\sum_{k=1}^{n}\left(2^{-2/n}\right)^k}_{\text{geometric}}$$

$$\sum_{k=0}^{N} ar^k = \frac{a(1-r^{N+1})}{1-r}$$

$$\parallel$$

$$a + \sum_{k=1}^{N} ar^k = \frac{a(1-r^{N+1})}{1-r}$$

$$\sum_{k=1}^{N} ar^k = \frac{a(1-r^{N+1})}{1-r} - a$$

$a=1$

$$\sum_{k=1}^{n} r^k = \frac{1-r^{n+1}}{1-r} - 1$$

$$\therefore \underbrace{\sum_{k=1}^{n}\left(2^{-2/n}\right)^k} = \frac{1-\left(2^{-\frac{2}{n}}\right)^{n+1}}{1-2^{-2/n}} - 1$$

$$\therefore \int_0^2 2^{-x}\,dx = \lim_{n\to\infty} \frac{2}{n}\left[\frac{1-\left(2^{-2/n}\right)^{n+1}}{1-2^{-2/n}}\right] - 1$$

$$= \boxed{\lim_{n\to\infty} \frac{2}{n}\left[\frac{1-2^{-2-2/n}}{1-2^{-2/n}} - 1\right]}$$

$\frac{0}{0}$ as $n \to \infty$

$$\boxed{\approx 1.08}$$